

ĐẠO HÀM HÀM SỐ MŨ-LOGARIT

$$\begin{aligned}
 1) (u^\alpha)' &= \alpha u^{\alpha-1} u' \rightarrow \begin{cases} (x^\alpha)' = \alpha x^{\alpha-1} \\ (\sqrt[n]{u})' = \frac{u'}{n\sqrt[n]{u^{n-1}}} \end{cases} & 2) (a^u)' &= u' a^u \ln a \rightarrow \begin{cases} (a^x)' = a^x \ln a \\ (e^u)' = u' e^u \\ (e^x)' = e^x \end{cases} \\
 3) (\log_a u)' &= \frac{u'}{u \ln a} \rightarrow \begin{cases} (\log_a x)' = \frac{1}{x \ln a} \\ (\ln u)' = \frac{u'}{u} \\ (\ln x)' = \frac{1}{x} \end{cases} & \text{Chú ý : 4) } (u^v)' &= u^v \cdot (v \ln u)' \text{ (Tổng quát của (1) và (2))}
 \end{aligned}$$

A. CÁC VÍ DỤ MINH HỌA

Ví dụ 1: Tính đạo hàm của các hàm số sau:

$$\begin{aligned}
 1) y &= \sqrt[3]{x+\sqrt{x}} & 2) y &= \sqrt{e^x} + e^{3x-1} - 5^{\cos x + \sin x} & 3) y &= (x^2 - 2x + 2)e^x \\
 4) y &= \ln(x^2 + 1) + \log_2(x^2 - x + 1) & 5) y &= \sqrt[3]{\ln^2 x} & 6) y &= \log_2\left(\frac{x-4}{x+4}\right) \\
 7) y &= \log\left(\frac{1-\sqrt{x}}{2\sqrt{x}}\right) & 8) y &= \frac{\ln x}{x} + \frac{1-\ln x}{1+\ln x} & 9) y &= \frac{\ln(2x-1)}{\sqrt{2x-1}} \\
 10) y &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & 11) y &= \ln(x + \sqrt{1+x^2}) + \log_3(\sin 2x) & 12) y &= \log_x(2x+1) & 13) y &= (2x-1)^{x+1}
 \end{aligned}$$

Giải:

$$\begin{aligned}
 1) y &= \sqrt[3]{x+\sqrt{x}} \Rightarrow y' = \frac{1 + \frac{1}{2\sqrt{x}}}{3\sqrt[3]{(x+\sqrt{x})^2}} = \frac{2\sqrt{x}+1}{6\sqrt{x}\sqrt[3]{(x+\sqrt{x})^2}} \quad (\text{áp dụng công thức } (\sqrt[n]{u})' = \frac{u'}{n\sqrt[n]{u^{n-1}}}) \\
 2) y &= \sqrt{e^x} + e^{3x-1} - 5^{\cos x + \sin x} \\
 \Rightarrow y' &= \frac{e^x}{2\sqrt{e^x}} + 3e^{3x-1} - (\tau \sin x + \cos x) \cdot 5^{\cos x + \sin x} \ln 5 = \frac{\sqrt{e^x}}{2} + 3e^{3x-1} + (\sin x - \cos x) \cdot 5^{\cos x + \sin x} \ln 5 \\
 3) y &= (x^2 - 2x + 2)e^x \Rightarrow y' = (2x - 2)e^x + (x^2 - 2x + 2)e^x = x^2 e^x \\
 4) y &= \ln(x^2 + 1) + \log_2(x^2 - x + 1) \Rightarrow y' = \frac{2x}{x^2 + 1} + \frac{2x-1}{(x^2 - x + 1)\ln 2} \\
 5) y &= \sqrt[3]{\ln^2 x} \Rightarrow y' = \frac{2 \cdot (\ln x) \cdot \frac{1}{x}}{3\sqrt[3]{\ln^4 x}} = \frac{2}{3x\sqrt[3]{\ln x}} \\
 6) y &= \log_2\left(\frac{x-4}{x+4}\right) \Rightarrow y' = \frac{\frac{8}{(x+4)^2}}{\left(\frac{x-4}{x+4}\right)\ln 2} = \frac{8}{(x^2 - 16)\ln 2}
 \end{aligned}$$

$$7) y = \log\left(\frac{1-\sqrt{x}}{2\sqrt{x}}\right) \Rightarrow y' = \frac{\left(\frac{1-\sqrt{x}}{2\sqrt{x}}\right)'}{\frac{1-\sqrt{x}}{2\sqrt{x}} \ln 10} = \frac{-\frac{1}{2\sqrt{x}} \cdot 2\sqrt{x} - \frac{1}{\sqrt{x}} \cdot (1-\sqrt{x})}{\frac{1-\sqrt{x}}{2\sqrt{x}} \ln 10} = \frac{-1 - \frac{1-\sqrt{x}}{\sqrt{x}}}{\frac{1-\sqrt{x}}{2\sqrt{x}} \ln 10} = \frac{1}{2x(\sqrt{x}-1) \ln 10}$$

$$8) y = \frac{\ln x}{x} + \frac{1-\ln x}{1+\ln x} \Rightarrow y' = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} + \frac{-\frac{1}{x}(1+\ln x) - \frac{1}{x}(1-\ln x)}{(1+\ln x)^2} = \frac{1-\ln x}{x^2} + \frac{-2}{x(1+\ln x)^2}$$

$$9) y = \frac{\ln(2x-1)}{\sqrt{2x-1}} \Rightarrow y' = \frac{\frac{2}{2x-1} \cdot \sqrt{2x-1} - \frac{1}{\sqrt{2x-1}} \cdot \ln(2x-1)}{2x-1} = \frac{2-\ln(2x-1)}{(2x-1)\sqrt{2x-1}}$$

$$10) y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow y' = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

$$11) y = \ln(x + \sqrt{1+x^2}) + \log_3(\sin 2x) \Rightarrow y' = \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} + \frac{2 \cos 2x}{\sin 2x \ln 3} = \frac{1}{\sqrt{1+x^2}} + \frac{2 \cot 2x}{\ln 3}$$

$$12) y = \log_x(2x+1) = \frac{\ln(2x+1)}{\ln x} \Rightarrow y' = \frac{\frac{2}{2x+1} \ln x - \frac{1}{x} \ln(2x+1)}{\ln^2 x} = \frac{2x \ln x - (2x+1) \ln(2x+1)}{x(2x+1) \ln^2 x}$$

$$13) y = (2x-1)^{x+1} \Rightarrow \ln y = \ln(2x-1)^{x+1} = (x+1) \ln(2x-1) \quad (*)$$

$$\Rightarrow \frac{y'}{y} = \ln(2x-1) + \frac{2(x+1)}{2x-1} \quad (\text{đạo hàm 2 vế của } (*))$$

$$\Rightarrow y' = \left[\ln(2x-1) + \frac{2(x+1)}{2x-1} \right] \cdot (2x-1)^{x+1}$$

Ví dụ 2: Chứng minh các đẳng thức sau:

1) $y'' + 2y' + 2y = 0$ với $y = e^{-x} \sin x$

2) $xy' + 1 = e^y$ với $y = \ln\left(\frac{1}{1+x}\right)$

3) $xy' = y(y \ln x - 1)$ với $y = \frac{1}{1+x+\ln x}$

4) $y + xy' + x^2 y'' = 0$ với $y = \sin(\ln x) + \cos(\ln x)$

5) $2x^2 y' = x^2 y^2 + 1$ với $y = \frac{1+\ln x}{x(1-\ln x)}$

6) $2y = xy' + \ln y'$ với $y = \frac{x^2}{2} + \frac{1}{2} x \sqrt{x^2+1} + \ln \sqrt{x + \sqrt{x^2+1}}$

Giải: 1) $y'' + 2y' + 2y = 0$ với $y = e^{-x} \sin x$

Ta có: $y = e^{-x} \sin x \Rightarrow \begin{cases} y' = -e^{-x} \sin x + e^{-x} \cos x = e^{-x} (\cos x - \sin x) \\ y'' = -e^{-x} (\cos x - \sin x) + e^{-x} (-\sin x - \cos x) = -2e^{-x} \cos x \end{cases}$

$$\Rightarrow y'' + 2y' + 2y = -2e^{-x} \cos x + 2e^{-x} (\cos x - \sin x) + 2e^{-x} \sin x = 0 \quad (\text{đpcm})$$

2) $xy' + 1 = e^y$ với $y = \ln\left(\frac{1}{1+x}\right)$

Ta có: $y = \ln\left(\frac{1}{1+x}\right) \Rightarrow y' = \frac{-\frac{1}{(1+x)^2}}{\frac{1}{1+x}} = \frac{-1}{1+x} \Rightarrow \begin{cases} xy' + 1 = \frac{-x}{1+x} + 1 = \frac{1}{1+x} \\ e^y = e^{\ln\left(\frac{1}{1+x}\right)} = \frac{1}{1+x} \end{cases} \Rightarrow xy' + 1 = e^y \quad (\text{đpcm})$

3) $xy' = y(y \ln x - 1)$ với $y = \frac{1}{1+x+\ln x}$. Ta có: $y = \frac{1}{1+x+\ln x} \Rightarrow y' = \frac{-\left(1+\frac{1}{x}\right)}{(1+x+\ln x)^2} = \frac{-(1+x)}{x(1+x+\ln x)^2}$

$\Rightarrow \begin{cases} xy' = \frac{-(1+x)}{(1+x+\ln x)^2} \\ y(y \ln x - 1) = \frac{1}{1+x+\ln x} \left(\frac{\ln}{1+x+\ln x} - 1 \right) = \frac{-(1+x)}{(1+x+\ln x)^2} \end{cases} \Rightarrow xy' = y(y \ln x - 1) \quad (\text{đpcm})$

4) $y + xy' + x^2 y'' = 0$ với $y = \sin(\ln x) + \cos(\ln x)$

Ta có: $y = \sin(\ln x) + \cos(\ln x) \Rightarrow \begin{cases} y' = \frac{1}{x} \cos(\ln x) - \frac{1}{x} \sin(\ln x) = \frac{\cos(\ln x) - \sin(\ln x)}{x} \\ y'' = \frac{\left[-\frac{1}{x} \sin(\ln x) - \frac{1}{x} \cos(\ln x) \right] x - [\cos(\ln x) - \sin(\ln x)]}{x^2} = \frac{-2\cos(\ln x)}{x^2} \end{cases}$

$\Rightarrow y + xy' + x^2 y'' = \sin(\ln x) + \cos(\ln x) + \cos(\ln x) - \sin(\ln x) - 2\cos(\ln x) = 0 \quad (\text{đpcm})$

5) $2x^2 y' = x^2 y^2 + 1$ với $y = \frac{1+\ln x}{x(1-\ln x)}$

Ta có: $y' = \frac{\frac{1}{x} \cdot x(1-\ln x) - \left[1-\ln x + x \cdot \left(-\frac{1}{x} \right) \right] (1+\ln x)}{x^2 (1-\ln x)^2} = \frac{1-\ln x + \ln x (1+\ln x)}{x^2 (1-\ln x)^2} = \frac{1+\ln^2 x}{x^2 (1-\ln x)^2}$

$\Rightarrow \begin{cases} 2x^2 y' = 2x^2 \cdot \frac{1+\ln^2 x}{x^2 (1-\ln x)^2} = \frac{2(1+\ln^2 x)}{(1-\ln x)^2} \\ x^2 y^2 + 1 = x^2 \cdot \frac{(1+\ln x)^2}{x^2 (1-\ln x)^2} + 1 = \frac{(1+\ln x)^2}{(1-\ln x)^2} + 1 = \frac{2(1+\ln^2 x)}{(1-\ln x)^2} \end{cases} \Rightarrow 2x^2 y' = x^2 y^2 + 1 \quad (\text{đpcm}).$

6) $2y = xy' + \ln y'$ với $y = \frac{x^2}{2} + \frac{1}{2} x \sqrt{x^2+1} + \ln \sqrt{x+\sqrt{x^2+1}}$

Ta có: $y' = x + \frac{1}{2} \left(\sqrt{x^2+1} + x \cdot \frac{x}{\sqrt{x^2+1}} \right) + \frac{1+\frac{x}{\sqrt{x^2+1}}}{\sqrt{x+\sqrt{x^2+1}}}$

$$= x + \frac{2x^2+1}{2\sqrt{x^2+1}} + \frac{x+\sqrt{x^2+1}}{2(x+\sqrt{x^2+1})\sqrt{x^2+1}} = x + \frac{2x^2+1}{2\sqrt{x^2+1}} + \frac{1}{2\sqrt{x^2+1}} = x + \frac{2(x^2+1)}{2\sqrt{x^2+1}} = x + \sqrt{x^2+1}$$

$\Rightarrow \begin{cases} xy' + \ln y' = x(x + \sqrt{x^2+1}) + \ln(x + \sqrt{x^2+1}) = x^2 + x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1}) \\ 2y = x^2 + x\sqrt{x^2+1} + 2\ln \sqrt{x+\sqrt{x^2+1}} = x^2 + x\sqrt{x^2+1} + \ln(x + \sqrt{x^2+1}) \end{cases} \Rightarrow 2y = xy' + \ln y' \quad (\text{đpcm})$

B. BÀI LUYỆN

Bài 1: Tính đạo hàm của các hàm số sau:

1) $y = \sqrt[3]{x^2 - x + 1}$

2) $y = (2x + 1)e^{3x-1}$

3) $y = xe^{\sqrt{x} - \frac{1}{3}x}$

4) $y = \frac{2^x}{x^2 - 2x + 2}$

5) $y = e^{3x-1} \cdot \cos 2x$

6) $y = (\sin x - \cos x)e^{2x}$

7) $y = (1 + \ln x) \ln x$

8) $y = \frac{\ln(x-1)}{x+1}$

9) $y = e^{2x} \ln(\cos x)$

10) $y = x^2 \ln \sqrt{x^2 + 1}$

11) $y = (x^2 + x) \log_2(2^x + e^{-x} - x)$

12) $y = \ln[\sin(3^x + 1)]$

Bài 2: Chứng minh các đẳng thức sau:

1) $xy' = (1 - x^2)y$ với $y = xe^{-\frac{x^2}{2}}$

2) $y' - y = e^x$ với $y = (x+1)e^x$

3) $y''' - 13y' - 12y = 0$ với $y = e^{4x} + 2e^{-x}$

4) $y' \cos x - y \sin x - y'' = 0$ với $y = e^{\sin x}$

5) $y'' - 2y' + y = e^x$ với $y = \frac{1}{2}x^2e^x$

6) $y' = \frac{2xy}{x^2 + 1} + e^x(x^2 + 1)$ với $y = (x^2 + 1)(e^x + 2013)$